

For this hw, recall the variation of parameters formula:

Let  $y_1, y_2$  be linearly independent solutions to

$$y'' + a_1(x)y' + a_0(x)y = 0$$

Then a particular solution to

$$y'' + a_1(x)y' + a_0(x)y = b(x)$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx \quad \text{and} \quad v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}$$

①(a) Solve  $y'' + y = \sec(x)$

Step 1: Solve  $y'' + y = 0$ .

The characteristic polynomial is  $r^2 + 1 = 0$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \frac{\pm \sqrt{-4}}{2} = \frac{\pm \sqrt{4} \sqrt{-1}}{2} = \pm \frac{2}{2} i = \pm i$$

$0 \pm i$

Thus,  $y_1 = e^{0x} \cos(x) = \cos(x)$

$y_2 = e^{0x} \sin(x) = \sin(x)$ .

And

$$y_h = c_1 \cos(x) + c_2 \sin(x).$$

Step 2: Find  $y_p$  for  $y'' + y = \sec(x)$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos^2(x) - (-\sin^2(x)) = 1$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-\sin(x) \sec(x)}{1} dx = - \int \frac{\sin(x)}{\cos(x)} dx$$

$$= - \int \tan(x) dx = - \ln |\sec(x)|$$

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{\cos(x) \sec(x)}{1} dx = \int \cos(x) \cdot \frac{1}{\cos(x)} dx$$
$$= \int 1 dx = x$$

So,

$$y_p = v_1 y_1 + v_2 y_2 = -\ln|\sec(x)| \cdot \cos(x) + x \sin(x)$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 \cos(x) + c_2 \sin(x) - \ln|\sec(x)| \cdot \cos(x) + x \sin(x)$$

This would be defined on any interval that  $\sec(x)$  is defined on. For example, when  $-\frac{\pi}{2} < x < \frac{\pi}{2}$  or  $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ .

①(b) Solve  $y'' + y = \sin(x)$

Step 1: Solve  $y'' + y = 0$ .

The characteristic polynomial is  $r^2 + 1 = 0$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \frac{\pm \sqrt{-4}}{2} = \frac{\pm \sqrt{4} \sqrt{-1}}{2} = \pm \frac{2}{2} i = \pm i$$

$0 \pm i$

Thus,  $y_1 = e^{0x} \cos(x) = \cos(x)$

$y_2 = e^{0x} \sin(x) = \sin(x)$ .

And

$$y_h = c_1 y_1 + c_2 y_2 = c_1 \cos(x) + c_2 \sin(x).$$

Step 2: Find  $y_p$  for  $y'' + y = \sin(x)$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos^2(x) - (-\sin^2(x)) = 1$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-\sin(x) \sin(x)}{1} dx = - \int \sin^2(x) dx$$

$$= - \int \left( \frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx = -\frac{1}{2}x - \frac{1}{4} \sin(2x)$$

$$\text{and } v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{\cos(x) \sin(x)}{1} dx$$

$$= \int \cos(x) \sin(x) dx = \int u du = \frac{u^2}{2} = \frac{1}{2} \sin^2(x)$$

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x) dx \end{aligned}$$

So,

$$y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{2}x - \frac{1}{4}\sin(2x)\right) \cos(x) + \frac{1}{2}\sin^2(x) \sin(x)$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 \cos(x) + c_2 \sin(x) - \frac{1}{2}x \cos(x) - \frac{1}{4}\sin(2x)\cos(x) + \frac{1}{2}\sin^2(x)\sin(x)$$

This solution is defined for all  $x$ , i.e. the interval is  $I = (-\infty, \infty)$

① (c1) Solve  $y'' - 9y = \frac{9x}{e^{3x}}$

Step 1: Solve  $y'' - 9y = 0$

The characteristic equation is  $r^2 - 9 = 0$

The roots are  $r = \pm 3$ .

Let  $y_1 = e^{3x}$ ,  $y_2 = e^{-3x}$ .

So,  $y_h = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-3x}$ .

Step 2: Find  $y_p$  for  $y'' - 9y = \frac{9x}{e^{3x}}$

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3e^{3x}e^{-3x} - 3e^{-3x}e^{3x}$$

$e^{3x-3x} = e^0 = 1$

$$= -3 - 3 = -6$$

So,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-e^{-3x} \cdot \frac{9x}{e^{3x}}}{-6} dx = \frac{9}{6} \int x e^{-6x} dx$$
$$= \frac{9}{6} \left( -\frac{1}{6} x e^{-6x} - \int -\frac{1}{6} e^{-6x} dx \right) = -\frac{9}{36} x e^{-6x} + \int \frac{9}{36} e^{-6x} dx$$

$u = x \quad du = dx$   
 $dv = e^{-6x} dx \quad v = -\frac{1}{6} e^{-6x}$

$$= -\frac{9}{36} x e^{-6x} - \frac{9}{36} \cdot \frac{1}{6} e^{-6x}$$
$$= -\frac{1}{4} x e^{-6x} - \frac{1}{24} e^{-6x}$$

and

$$v_2 = \int \frac{y_1 \cdot b(x)}{w(y_1, y_2)} dx = \int \frac{e^{3x} \cdot \frac{9x}{e^{3x}}}{w(y_1, y_2)} dx = \int \frac{9x}{-6} dx$$
$$= -\frac{3}{2} \cdot \frac{x^2}{2} = -\frac{3}{4} x^2$$

So,

$$y_p = v_1 y_1 + v_2 y_2 = \left( -\frac{1}{4} x e^{-6x} - \frac{1}{24} e^{-6x} \right) \cdot e^{3x}$$
$$+ \left( -\frac{3}{4} x^2 \right) \cdot e^{-3x}$$
$$= -\frac{1}{4} x e^{-3x} - \frac{1}{24} e^{-3x} - \frac{3}{4} x^2 e^{-3x}$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{4} x e^{-3x} - \frac{1}{24} e^{-3x} - \frac{3}{4} x^2 e^{-3x}$$

this function is defined for all  $x$ , so  
the interval would be  $I = (-\infty, \infty)$

①(d) Solve  $y'' + 3y' + 2y = \frac{1}{1+e^x}$

Step 1: Solve  $y'' + 3y' + 2y = 0$ .

The characteristic equation is  $r^2 + 3r + 2 = 0$ .

This factors as  $(r+2)(r+1) = 0$

So, the roots are  $r = -1, -2$ .

Let  $y_1 = e^{-x}$ ,  $y_2 = e^{-2x}$

Then,  $y_h = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{-2x}$

step 2: Find  $y_p$  for  $y'' + 3y' + 2y = \frac{1}{1+e^x}$

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = (e^{-x})(-2e^{-2x}) - (-e^{-x})(e^{-2x})$$

$$= -2e^{-3x} + e^{-3x} = -e^{-3x}$$

So,

$$V_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-e^{-2x} \cdot \frac{1}{1+e^x}}{-e^{-3x}} dx = \int \frac{e^{3x-2x}}{1+e^x} dx$$

$$= \int \frac{e^x}{1+e^x} dx = \int \frac{1}{1+u} du = \ln|1+u| = \ln|1+e^x|$$

$\uparrow$   $u = 1+e^x \quad du = e^x dx$  (1+e^x > 0)  $\downarrow = \ln(1+e^x)$



and

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{e^{-x} \cdot \frac{1}{1+e^x}}{-e^{-3x}} dx = - \int \frac{e^{3x-x}}{1+e^x} dx$$

$$\int \frac{-e^{2x}}{1+e^x} dx = \int \left( \frac{e^x}{1+e^x} - e^x \right) dx$$

see  $v_1$  calculation

$$= \ln(1+e^x) - e^x$$

$$\frac{-e^{2x}}{1+e^x} = \frac{e^x}{1+e^x} - e^x$$

Why!

$$\begin{aligned} \frac{e^x}{1+e^x} - e^x &= \frac{e^x - e^x(1+e^x)}{1+e^x} \\ &= \frac{e^x - e^x - e^{2x}}{1+e^x} \\ &= \frac{-e^{2x}}{1+e^x} \end{aligned}$$

Thus,

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 = \ln(1+e^x) \cdot e^{-x} + (\ln(1+e^x) - e^x) \cdot e^{-2x} \\ &= e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x) - e^{-x} \end{aligned}$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x) - e^{-x}$$

This is defined on  $I = (-\infty, \infty)$

①(e) Solve  $y'' + 3y' + 2y = \sin(e^x)$

Step 1: Solve  $y'' + 3y' + 2y = 0$

The characteristic equation is  $r^2 + 3r + 2 = 0$

This factors as  $(r+2)(r+1) = 0$

The roots are  $r = -2, -1$

Thus,  $y_1 = e^{-2x}$ ,  $y_2 = e^{-x}$

And  $y = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} + c_2 e^{-x}$

Step 2: Find  $y_p$  for  $y'' + 3y' + 2y = \sin(e^x)$

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = (e^{-2x})(-e^{-x}) - (e^{-x})(-2e^{-2x})$$
$$= -e^{-3x} + 2e^{-3x}$$
$$= e^{-3x}$$

So,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-e^{-x} \sin(e^x)}{e^{-3x}} dx$$

$$= - \int e^{2x} \sin(e^x) dx = - \int t \sin(t) dt$$

$t = e^x$   
 $dt = e^x dx$

$e^{2x} = (e^x)(e^x)$   
 $= t dt$

$$= - \left[ -t \cos(t) - \int -\cos(t) dt \right]$$

$u = t$   
 $du = dt$   
 $dv = \sin(t) dt$   
 $v = -\cos(t)$

$$= t \cos(t) - \sin(t)$$

$$= e^x \cos(e^x) - \sin(e^x)$$

And

$$v_2 = \int \frac{y_1 b(x)}{w(y_1, y_2)} dx = \int \frac{e^{-2x} \sin(e^x)}{e^{-3x}} dx = \int e^x \sin(e^x) dx$$

$$= \int t \sin(t) dt = -t \cos(t) - \int -\cos(t) dt$$

$t = e^x$   
 $dt = e^x dx$

$u = t$   
 $du = dt$   
 $dv = \sin(t) dt$   
 $v = -\cos(t)$

$$= -t \cos(t) + \sin(t) = -e^x \cos(e^x) + \sin(e^x)$$

So,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= (e^x \cos(e^x) - \sin(e^x)) e^{-2x}$$

$$+ (-e^{-x} \cos(e^x) + \sin(e^x)) e^{-x}$$

$$= e^{-x} \cos(e^x) - e^{-2x} \sin(e^x) \\ - e^{-2x} \cos(e^x) + e^{-x} \sin(e^x)$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 e^{-2x} + c_2 e^{-x} \\ + e^{-x} \cos(e^x) - e^{-2x} \sin(e^x) \\ - e^{-2x} \cos(e^x) + e^{-x} \sin(e^x)$$

This solution is defined on  $I = (-\infty, \infty)$